1. Probability Theory
   1. Probability Theory as a Set of Outcome

* Terminology :
* **Experiment**: a probabilistic model, the output is not deterministic
* **Sample space(**: the set of all possible outcomes
* **Sample points( the elements of the sample space**
* **Events**: subsets of sample space
* Space:

Euclidean space / Linear space / Probability space / ….

* Deterministic: the output is determined, the fixed number, quantitative.

Probabilistic (Stochastic): the output is not determined.

Def. 1.1 **An event** is an outcome or a collection of outcomes. It is **a set**, and hence we use set notation to denote an event, , called set A is a subset of

* Roll a fair die ,
* Roll an odd die
* Roll a fair die: Is this 🡪 No! there is no “7” outcome of the experiment
  1. Set Theory

Def. 1.4.

1. The sets A and B are equal to sets or identical sets iff A and B have the same elements. We denote equality by writing A=B
2. A is included in B or A is a subset of B iff implies In such cases, we write

Prop. 1.6.

1. iff
2. If then

* Union --> logic as “or”
* Intersection 🡪 logic as “and”
* Complement,
* Relative complement (difference) .

Prop.1.7.

1. then

* Proof

1. By Contradiction: proof the claim is not correct.
2. and and
3. If , by assumption
4. b) is contradict to a) since 🡪 the proof is complete. QED
5. First we prove direction i.e., if

2.1) we know and .

2.2) Since if then (the first part of this proposition) , and . Hence

2.3) Now we prove direction i.e.,

2.4) implies or or

2.5) Hence

2.6) 2.5) implies (\*remember the definition of Union)

* De Morgan’s law

1. Not (A and B ) < -- > not (A) or not( B)
2. Not( A or B) < -- > not(A) and not(B)

* The null set : the set which is no elements.

1. In set theory, no definition of the null element.
2. The facts of

-If the set A and B are no common elements, then

-

-

-

-Since , this implies and . Hence the

* A set A and B is disjoint if .
* Question: the number of subsets given a set

1) Let how many subsets of

-{1},{2},{3},{1,2},{1,3},{2,3},[1,2,3] and 🡪 8

2) Prove if # of = n, then # of subset of is

* 1. Prob. Space and the Prob. Measure

Axiom 1. Given an experiment, there exists a sample space , representing **the totality** **of possible outcome** of the experiment and a collection, of subsets, A, of called events

* **Sample space(**: the set of all possible outcomes
* **Events**: a collection of subsets of sample space **(**
* Axiom :

A statement that is taken to be true, to serve as a premise or stating point for further reasoning and arguments.

* We do not need to prove the axiom.
* 1) It is possible to draw a [straight line](https://en.wikipedia.org/wiki/Straight_line) from any point to any other point.
* 2)

Axiom 2. each event A in there can be assigned a nonnegative number P(A) such that

Lemma 1.9.

* Proof:

Implies .QED

Lemma 1.10. If A and B are two arbitrary events in the sample space , then

* Proof:

Hence

And  
.

Hence,

which gives to

* 1. Algebra of Sets and Pro. Space
* Measurable / countable / uncountable…
* # of events is **infinite**, then something is weird…later or you may see math. Textbook for the measurable theory. It is to spend too much energy to learn…!! But should be accounted for….In this course just to see a glimpse of the measure theory….
* In order to define the probability we need the following questions be considered.

+ given a sample space as, interval ,

1. How many real numbers in
2. How many rational numbers in
3. How many irrational numbers in
4. Is it possible to define the probability
5. How about ?
6. Is it possible to define the probability of the infinite unions of disjoint events?

* All these questions in answered in “Measure Theory”, “Number Theory”, or, in “Real analysis” .
* We should learn “Measure Theory”, to study the probability. But as YOU EXPECT , let us just have a glimpse of all. **WE DO NOT NEED SPEND LOTS OF ENERGY.**

Def.1.12. An algebra, , **is a set of subsets of**  such that the following hold:

1. implies
2. implies

* Difference between event and algebra

1. **An event** is an outcome or a collection of outcomes. It is **a set**.

1. An algebra, , **is a set of subsets of**

* Example

- ={1,2}, set is an algebra

- ={1,2}, set is an algebra

- ={1,2}, set is not an algebra since

- ={1,2,3}, set is not an algebra

Prop. 1.13. If and

2. and

* Proof:

1. Since is an algebra, . Hence .In addition, , implies
2. By definition
3. and .QED

Exam 1.14 / 1.15 / Remark 1.16

Def. 1.17. A class of subsets of is a , denoted **,** if it is an algebra and if it is also closed under countable unions, i.e.,

Axiom 3. Let , of subsets of and a probability measure defined on elements of Then, if is a countable of disjoint sets, i.e., , the probability of the union is found by

* Axiom 2 is replace to Axiom 3.

Def. 1.19. If is the set containing all possible outcomes of an experiment,  **is a**  of the subsets of and is a probability measure on **,** then the triple

is called a **probability space.**

* An example of uncountable operations.

Let construct of space by defining probabilities for intervals as being the length of the intervals of which points are all equally possible. Define probability of any interval as .

For example, .

Let define an – algebra as

the set of all countable operations like (a,b)}

1. The probability of any singletons :

Since , which is in **.**

Hence the probability

Hence

1. Uncountable set operation

Let forming the union of all points in

Thus

Now the left of hand side

Right hand side

Which implies

Hence uncountable set operation is not allowed in probability space.

* 1. Key Concepts in Probability Theory

Def. 1.24. A conditional probability is the probability of the occurrence of an event subject to the hypothesis that another event has occurred

* Joint probability

The collection of events .

Joint probability : The probability of

* Marginal probability: Let the sample space be partitioned into two different families of disjoint sets, and

The marginal probability of

Def. 1.24. A conditional probability is the probability of the occurrence of an event

Subject to the hypothesis that **another event has occurred**.

The probability of the event B given A = The conditional probability of B given A

* (Axiom/ definition..)
* Properties of the conditional probabilities

1. If , then the marginal probability of B is
2. **Bayes’s rule** :

* Independence (statistical independence):

Two events A and B are independent if

Which is equivalent to

Ex. 1.25

Remark 1.26

* Orthogonality or mutual exclusivity or disjoint:

Two events, A and B , are orthogonal / mutually exclusive / disjoint if

* Remember in linear system: Independent / orthogonal…

(in linear system:

- Two vectors are orthogonal 🡪 two vectors are independent

- Two vector are independent 🡪 not always independent.